Analytical solutions for bending and buckling of functionally graded nanobeams based on the nonlocal Timoshenko beam theory

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In this paper, static bending and buckling of a functionally graded (FG) nanobeam are examined based on the nonlocal Timoshenko and Euler–Bernoulli beam theory. This non-classical (nonlocal) nanobeam model incorporates the length scale parameter (nonlocal parameter) which can capture the small scale effect. The material properties of the FG nanobeam are assumed to vary in the thickness direction. The governing equations and the related boundary conditions are derived using the principal of the minimum total potential energy. The Navier-type solution is developed for simply-supported boundary conditions, and exact formulas are proposed for the deflections and the buckling load. The effects of nonlocal parameter, aspect ratio, various material compositions on the static and stability responses of the FG nanobeam are discussed. Some illustrative examples are also presented to verify the present formulation and solutions. Good agreement is observed. The results show that the new nonlocal beam model produces larger deflection and smaller buckling load than the classical (local) beam model.

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1. Introduction

Nanotechnology is able to create many new materials and devices with a vast range of applications, such as in medicine, electronics, biomaterials and energy production. Structural elements, such as beams and plates in micro or nanoscale range are commonly used as components in micro/nanoelectromechanical systems (MEMS/NEMS). In all these applications, the thickness of the micro- and nanostructures is typically on the order of micrometer and nanometer. Both experimental and simulation results have shown a significant size effect in mechanical properties when the dimensions of these structures become very small. For this reason, the size effect has a major role on static and dynamic behavior of micro, nanostructures and cannot be ignored. It is well-known that classical continuum mechanics does not account for such size effects in micro-, nano-scale structures. In order to overcome this problem, many higher order continuum (non-local) theories that contain additional material constants, such as the modified couple stress theory [1], the strain gradient theory [2], the micropolar theory [3], the nonlocal elasticity theory [4], and the surface elasticity [5], have been developed to characterize the size effect in micro, nano-scale structures by introducing an intrinsic length scale in the constitutive relations. Among these theories, the nonlocal elasticity theory, which was introduced by Eringen [6] to account for scale effect in elasticity, was used to study lattice dispersion of elastic waves, wave propagation in composites, dislocation mechanics, fracture mechanics and surface tension fluids. After this, Peddieson et al. [7] first applied the nonlocal continuum theory to the nanotechnology in which the static deformations of beam structures were obtained by using a simplified nonlocal beam model based on the nonlocal elasticity theory of Eringen [6].

In the classical (local) elasticity theory, the stress at a point depends only on the strain at the same point whereas in the nonlocal elasticity theory, the stress at a point is a function of strains at all points in the continuum. In this way, the nonlocal continuum theory contains information about long range forces between atoms, and the internal length scale is introduced into the constitutive equations simply as material parameter to capture the small scale effect. In this context, the application of the classical continuum theory to the analysis of nanostructures is not appropriate since the classical theories lack the accountability of the size effects arising from the small-scale. The application of nonlocal elasticity in micro and nanomaterials has received much attention among the nanotechnology community recently, and also the literature shows that the nonlocal elasticity theory is being increasingly used for reliable and quick analysis of nanostructures. For example, static [8–12], buckling [13–21], vibration [22–39], wave propagation [40–45] and thermo-mechanical [46–51] analysis of nanostructures are studied by several researchers. Most recently, Phadikar and Pradhan

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[52] introduced a variational formulation and the finite element analysis for the structural analysis of nanobeams and nanoplates based on the nonlocal stress gradient elasticity theory. Mustapha and Zhong [53] investigated the free vibration of an axially loaded non-prismatic single-walled carbon nanotube embedded in a two-parameter elastic medium by using the Bubnov–Galerkin method. Murmu and Adhikari [54] have developed a nonlocal double-elastic beam model, and applied it to investigate the small scale effect on the free vibration of the double-nanobeam system. Roque et al. [55] used the nonlocal elasticity theory of Eringen to study bending, buckling and free vibration of Timoshenko nanobeams by using a meshless method. Thai [56] and Thai and Vo [57] proposed nonlocal shear deformation beam theory for bending, buckling and vibration of homogeneous nanobeams based on the Eringen’s nonlocal elasticity theory.

Developments in the field of materials engineering lead to a new type of materials with smooth and continuous variation of the material properties that called functionally graded materials (FGMs). The mechanical and thermal response of materials with spatial gradients in composition and microstructure is of considerable interest in numerous technological areas such as tribology, optoelectronics, biomechanics, nanotechnology and high temperature technology. The graded transition in composition across an interface of two materials (for instance, metal and ceramics or polymer) can essentially reduce the thermal stresses and stress concentration at intersection with free surfaces. Also, FGMs will allow a tailored fit to a special purpose, i.e., with the static deflection not exceeding a specific level, or the buckling load not being less than a pre-specified level, or the natural frequency either exceeding or to being less than a pre-specified frequency. In view of these advantages, a number of investigations, dealing with static, buckling and dynamic characteristics of functionally graded (FG) structures, were published in the scientific literature [58–73]. With the rapid development of technology, functionally graded (FG) beams and plates have been started to use in micro/nanoelectromechanical systems (MEMS/NEMS), such as the components in the form of shape memory alloy thin films with a global thickness in micro- or nano-scale [74–76], electrically actuated MEMS devices [77–79], and atomic force microscopes (AFMs) [80]. Since the dimension of these structural devices typically falls below micron- or nanoscale in at least one direction, an essential feature triggered in these devices is that their mechanical properties such as Young’s modulus and flexural rigidity are size-dependent. So far, only a few works have been reported for FG nanobeams based on the nonlocal elasticity theory. Janghorban and Zare [81] investigated nonlocal free vibration axially FG nanobeams by using differential quadrature method. Eltaher et al. [82] studied free vibration of FG nanobeam based on the nonlocal Euler–Bernoulli beam theory. Free vibration of axially FG tapered nanorods has been examined based on the nonlocal elasticity theory by Şimşek [83].

Therefore, based on the above discussion there is a strong encouragement to understand the mechanical behavior of FG nanobeams in the design of nanodevices. This paper focuses on the static bending and the buckling of the FG nanobeam based on the nonlocal Timoshenko and Euler–Bernoulli beam theory. It should be stressed at this stage that Euler–Bernoulli beam theory neglects the effect of transverse shear strain because the assumption of plane cross-sections perpendicular to the axis of the beam remaining plane and perpendicular to the axis after deformation. This simple beam theory can give excellent solutions to the analysis of slender beams. However, when a beam is moderately deep, this theory needs some modifications to include the effect of transverse shear. Timoshenko beam theory, which takes into consideration the influences of transverse shear deformation, is more appropriate for the estimation of the mechanical behavior of short beams. The material properties of the FG nanobeam are assumed to vary in the thickness direction. The governing equations and the related boundary conditions are derived using the principal of the minimum total potential energy. The Navier-type solution is developed for simply-supported boundary conditions, and exact formulas are proposed for the deflections and the buckling load. The effects of nonlocal parameter, aspect ratio, various material compositions on the static and stability responses of the FG nanobeam are discussed. Some illustrative examples are also presented to verify the present formulation and solutions. Good agreement is observed.

2. Nonlocal elasticity theory

Nonlocal elasticity theory was first introduced by Eringen [6], and the stress field at a point \( x \) in an elastic continuum not only depends on the strain field at the same point but also on strains at all other points of the body. Therefore, the nonlocal stress tensor \( \sigma \) at point \( x \) is defined by:

\[
\sigma(x) = \int_k k(|x' - x|, \ell) T(x') dV(x')
\]

\[
T(x) = C(x) : e(x)
\]

where \( T(x') \) is the classical, macroscopic stress tensor at point \( x' \), \( k(|x' - x|, \ell) \) is the nonlocal modulus or attenuation function incorporating into constitutive equations the nonlocal effects at the reference point \( x' \) produced by local strain at the source \( x' \), \( C(x) \) is the fourth-order elasticity tensor, \( e(x) \) is the strain tensor, \( \ell \) is the material constant which is defined as \( \ell = e_0 a/(l) \) where \( e_0 \) is a constant appropriate to each material, \( a \) is an internal characteristics length (e.g., lattice parameter, granular distance) and \( l \) is an external characteristics length (e.g., crack length, wavelength). Since solving the integral constitutive Eq. (1) is difficult, a simplified equation of differential form is used as a basis of all nonlocal constitutive formulation [6]:

\[
(1 - \nu^2 \nabla^2) \sigma = T \quad \tau = e_0 a/\ell
\]

where \( \nabla^2 \) is the Laplacian operator. For a beam type structure, the nonlocal behavior can be neglected in the thickness direction. Thus, the solid constitutive relation takes the following form:

\[
\sigma_{xx} - (e_0 a/\ell) \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E e_{xx}
\]

\[
\sigma_{xz} - (e_0 a/\ell) \frac{\partial^2 \sigma_{xz}}{\partial x^2} = G e_{xz}
\]

where \( E \) is the elasticity modulus, \( G = 0.5E/(1 + \nu) \) is the shear modulus (where \( \nu \) is the Poisson’s ratio), \( \sigma_{xx} \) is the axial normal stress, \( \sigma_{xz} \) is the shear stress, \( e_{xx} \) is the axial strain and \( e_{xz} \) is the shear strain. When the nonlocal parameter is taken as \( e_0 a = 0 \), the constitutive relation of the local theory is obtained.

3. Functionally graded materials

Fig. 1 shows a functionally graded (FG) nanobeam of length \( L \), width \( b \), and thickness \( h \). The FG microbeam is subjected to the distributed transverse load \( q(x) \) and an axial compressive force \( P \). The \( x_1 = x, x_2 = y \) and \( x_3 = z \) coordinates are chosen along the length, width, and the thickness of FG nanobeam, respectively.

We assumed that the FG nanobeam is made of two different materials, and the effective material properties of the FG nanobeam (i.e., Young’s modulus \( E \), Poisson’s ratio \( \nu \) and shear modulus \( \mu \), \( \rho \)) vary continuously in the thickness direction (in the \( z \) direction). According to the rule of mixture, the effective material properties \( (P) \) can be expressed as:

\[
P = P_1 V_1 + P_2 V_2
\]
where $P_1$, $P_2$ are the effective material properties, $V_1$ and $V_2$ are the volume fractions of the first and the second material related by:

$$V_1 + V_2 = 1$$  \hspace{1cm} (7)

The effective material properties of the FG nanobeam are defined by the power-law form. The volume fraction of the second material is assumed by:

$$V_2(z) = \left(\frac{z}{h} + \frac{1}{2}\right)^k, \quad V_1(z) = 1 - \left(\frac{z}{h} + \frac{1}{2}\right)^k$$  \hspace{1cm} (8)

where $k$ is the non-negative parameter (power-law exponent) which dictates the material variation profile through the thickness of the beam. Using Eqs. (6) and (8), the effective material properties of the FG microbeam can be expressed as follows:

$$P(z) = (P_2 - P_1)\left(\frac{z}{h} + \frac{1}{2}\right)^k + P_1$$  \hspace{1cm} (9)

Eq. (9) can be written in terms of Young’s modulus and Poisson’s ratio as:

$$E(z) = (E_2 - E_1)\left(\frac{z}{h} + \frac{1}{2}\right)^k + E_1$$  \hspace{1cm} (9)

$$\nu(z) = (\nu_2 - \nu_1)\left(\frac{z}{h} + \frac{1}{2}\right)^k + \nu_1$$  \hspace{1cm} (10)

It is easily seen that $E = E_2$, $\nu = \nu_2$ when $z = + h/2$, and $E = E_1$, $\nu = \nu_1$ when $z = - h/2$. Fig. 2 shows the variation of Young’s modulus along the thickness.

4. The governing equations based on the nonlocal elasticity theory

4.1. Timoshenko beam theory (TBT)

Based on the Timoshenko beam theory, the axial displacement, $u$, and the transverse displacement of any point of the beam, $w$, are given as:

$$u_x(x,z) = u(x) - z\phi(x)$$  \hspace{1cm} (11)

$$u_y(x,z) = 0$$  \hspace{1cm} (12)

$$u_z(x,z) = w(x)$$  \hspace{1cm} (13)

where $u$ and $w$ are the axial and the transverse displacement of any point on the neutral axis, $\phi$ is the total bending rotation of the cross-sections at any point on the neutral axis. The nonzero strains of the Timoshenko beam theory are obtained as:

$$e_{xx} = u_x^0 - z\kappa_x, \quad e_{yy} = 0, \quad \kappa_x = \frac{d\phi}{dx}$$  \hspace{1cm} (14)

$$\gamma_{xz} = \frac{dw}{dx} - \phi$$  \hspace{1cm} (15)

The governing equations and the boundary conditions will be derived by using principal of the minimum total potential energy. The first variation of the strain energy is given as:

$$\delta U_{str} = \int_V \sigma_{ij}\delta e_{ij} dV = \int_V \left(\sigma_{xx}\delta e_{xx} + \sigma_{xz}\delta e_{xz}\right) dV$$

$$= \int_0^L \left(N\delta e_x^0 + M\delta \kappa_x + Q\delta \gamma_{xz}\right) dx$$  \hspace{1cm} (16)

where $U_{str}$ is the strain energy, $N$ is the axial normal force, $Q$ is the shear force and $M$ is the bending moment. These stress resultants are defined as

$$N = \int_A \sigma_{xx} dA, \quad Q = \int_A k_i\sigma_{xz} dA, \quad M = \int_A z\sigma_{yy} dA$$  \hspace{1cm} (17)

where $k_i$ is the shear correction factor. The first variation of the work done by the axial compressive force is given by:

$$\delta W_{ext} = \int_0^L \left(P \left(\frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx}\right) dx + \mathcal{V}\delta w_{10} + \overline{M}\delta \phi_{10}^{\text{e}}\right)$$

$$+ \int_0^L \left(q\delta w + f\delta u\right) dx$$  \hspace{1cm} (18)

where $P$ is the axial compressive force, $\mathcal{V}$ is the external shear force, $\overline{M}$ is the external bending moment, $f$ is the distributed transverse load. According to the minimum total potential energy principle, the first variation of the total potential energy must be zero. That is

$$\delta \Pi = \delta(U_{int} - W_{ext}) = 0$$  \hspace{1cm} (19)
where $\Pi$ is the total potential energy. Substituting Eqs. (16) and (18) into Eq. (19), integrating by parts and setting the coefficient $\dot{u}$, $\dot{w}$ and $\ddot{\phi}$ to zero lead to the following governing equations:

$$\frac{dN}{dx} + f = 0$$

$$\frac{dQ}{dx} - \frac{d}{dx}\left(\frac{d^2w}{dx^2}\right) + q = 0$$

$$\frac{dM}{dx} - Q = 0$$

Furthermore, the following boundary conditions at the edges of the beam (at $x = 0, L$) are obtained

- $N = \overline{N}$ or $u = \overline{u}$
- $Q = \overline{Q}$ or $w = \overline{w}$
- $M = \overline{M}$ or $\phi = \overline{\phi}$

By using Eqs. (4), (5), (14), (15) and (17), the force–strain and the moment–strain relations of the nonlocal Timoshenko beam theory can be obtained as follows:

$$N - (\varepsilon_0) \frac{d^2N}{dx^2} = A_{xx} \frac{du}{dx} - B_{xx} \frac{d\phi}{dx}$$

$$M - (\varepsilon_0) \frac{d^2M}{dx^2} = B_{xx} \frac{du}{dx} - D_{xx} \frac{d\phi}{dx}$$

$$Q - (\varepsilon_0) \frac{dQ}{dx} = k_{Ax} \left(\frac{dw}{dx} - \phi\right)$$

In the above equations, the following cross-sectional rigidities are defined:

$$A_{xx} = \int_A G(z) \frac{1}{dx^2} \, dA$$

$$D_{xx} = \int_A G(z) \frac{1}{dx^2} \, dA$$

The explicit expression of the nonlocal normal force can be obtained by substituting for the second derivative of $N$ from Eq. (20) into Eq. (26) as follows:

$$N = A_{xx} \frac{du}{dx} - B_{xx} \frac{d\phi}{dx} - (\varepsilon_0) \frac{d^2N}{dx^2}$$

Eliminating the shear force $Q$ between Eqs. (21) and (22) leads to the following equation:

$$\frac{d^2M}{dx^2} - \frac{d}{dx}\left(\frac{d^2w}{dx^2}\right) + q = 0$$

The explicit expression of the nonlocal bending moment can be obtained by substituting for the second derivative of $M$ from Eq. (32) into Eq. (27) as follows:

$$M = B_{xx} \frac{du}{dx} - D_{xx} \frac{d\phi}{dx} - (\varepsilon_0) \frac{d^2M}{dx^2}$$

By substituting for the second derivative of $Q$ from Eq. (21) into Eq. (28), the following relation for the nonlocal shear force is obtained:

$$Q = k_{Ax} \left(\frac{dw}{dx} - \phi\right) + (\varepsilon_0) \left(\frac{d^2w}{dx^2} - \frac{dq}{dx}\right)$$

Finally, the nonlocal governing equations in terms of the displacements can be obtained by substituting for $N$, $M$ and $Q$ from Eqs. (31), (33), and (34), respectively, into Eqs. (20)–(22) as follows:

$$A_{xx} \frac{d^2u}{dx^2} - B_{xx} \frac{d^2\phi}{dx^2} = (\varepsilon_0) \left(\frac{d^2f}{dx^2} - f\right)$$

$$k_{Ax} \left(\frac{d^2w}{dx^2} - \frac{d\phi}{dx}\right) - \frac{d}{dx}\left(\frac{d^2w}{dx^2}\right) + \frac{P(\varepsilon_0)}{\varepsilon_0} \frac{d^2w}{dx^2} = (\varepsilon_0) \frac{d^2q}{dx^2} - q$$

$$- B_{xx} \frac{d^2u}{dx^2} + D_{xx} \frac{d^2\phi}{dx^2} + k_{Ax} \left(\frac{dw}{dx} - \phi\right) = 0$$

4.2. Euler–Bernoulli beam theory (EBT)

The displacement field of Euler–Bernoulli beam theory can be given as:

$$u(x, z) = u(x) - \frac{z}{2} \frac{d^2w(x)}{dx}$$

$$u(y, z) = 0$$

$$u(z, z) = w(x)$$

The axial strain $\varepsilon_{xx}$ of Euler–Bernoulli beam theory is expressed as:

$$\varepsilon_{xx} = \varepsilon_0 - \kappa_s, \quad \varepsilon_{xx} = \frac{du}{dx}, \quad \kappa_s = \frac{d^2w}{dx^2}$$

With the similar procedure that was applied for Timoshenko beam theory, the force–strain and the moment–strain relations of the nonlocal Euler–Bernoulli beam theory can be obtained as follows:

$$N = A_{xx} \frac{du}{dx} - B_{xx} \frac{d\phi}{dx} - (\varepsilon_0) \frac{d^2N}{dx^2}$$

$$M = B_{xx} \frac{du}{dx} - D_{xx} \frac{d\phi}{dx} - (\varepsilon_0) \frac{d^2M}{dx^2}$$

Note that shear force is obtained zero for Euler–Bernoulli beam theory since the shear strain is neglected in this theory. Also, nonlocal axial normal force and bending moment can be found as:

$$N = A_{xx} \frac{du}{dx} - B_{xx} \frac{d\phi}{dx} - (\varepsilon_0) \frac{d^2N}{dx^2}$$

$$M = B_{xx} \frac{du}{dx} - D_{xx} \frac{d\phi}{dx} - (\varepsilon_0) \frac{d^2M}{dx^2}$$

The nonlocal governing equations in terms of the displacements can be obtained by substituting for $N$, $M$ from Eqs. (44) and (45), respectively, into Eqs. (20)–(22) as follows:

$$A_{xx} \frac{d^2u}{dx^2} - B_{xx} \frac{d^2\phi}{dx^2} = (\varepsilon_0) \frac{d^2f}{dx^2} - f$$

$$B_{xx} \frac{d^2u}{dx^2} - D_{xx} \frac{d^2\phi}{dx^2} - \frac{d}{dx}\left(\frac{d^2w}{dx^2}\right) + \frac{P(\varepsilon_0)}{\varepsilon_0} \frac{d^2w}{dx^2} = (\varepsilon_0) \frac{d^2q}{dx^2} - q$$

5. Analytical solution for static bending and buckling of a simply-supported FG nanobeam

In this section, the governing equations are analytically solved for static bending and buckling of a simply-supported FG nanobeam. The Navier solution procedure is used to determine the analytical solutions for the simply-supported boundary conditions. For this purpose, the displacement functions are expressed as product of undetermined coefficients and known trigonometric functions to satisfy the governing equations and the conditions at $x = 0, L$. The following displacement fields are assumed to be of the form [9,27]:

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u(x) = \sum_{n=1}^{N} U_n \cos n\pi x  \quad (48)

w(x) = \sum_{n=1}^{N} W_n \sin n\pi x  \quad (49)

\phi(x) = \sum_{n=1}^{N} \phi_n \cos n\pi x  \quad (50)

where \((U_n, W_n, \phi_n)\) are the unknown Fourier coefficients to be determined for each \(n\) value, and \(x = \pi n/L\). It should be noted that \(G_n\) is identically zero for Euler–Bernoulli beam theory.

5.1. Static bending

For the static bending problem, the axial compressive force \(P\) and the external axial load \(f\) are set to zero. Also, the applied transverse load \(q\) is expanded in Fourier series as

\[ q(x) = \sum_{n=1}^{N} q_n \sin n\pi x \quad (51) \]

\[ Q_n = \frac{2}{L} \int_0^L q(x) \sin n\pi x \, dx \quad (52) \]

where \(Q_n\) are the Fourier coefficients, and are given for uniform load as follows [9, 27]:

Uniform load, \(q(x) = q_0\): \(Q_n = \frac{4q_0}{n\pi} \quad n = 1, 3, 5, \ldots \) (53)

Point load, \(q(x) = p_0 \delta(x-x_p)\): \(Q_n = \frac{2p_0}{L} n\pi \sin n\pi x_p \quad n = 1, 2, 3, \ldots \) (54)

where \(q_0\) is the intensity of the uniformly distributed load, \(\delta(\cdot)\) is the Dirac delta function, \(p_0\) is the magnitude of the point load, \(x_p\) is the application position of the point load. When the point load is acted on the midspan of the beam, namely, \(q(x) = p_0 \delta(x-1/L)\), the Fourier coefficients take the following form

\[ Q_n = \frac{2p_0}{L} \sin \frac{n\pi}{2} \quad n = 1, 2, 3, \ldots \] (55)

5.1.1. Timoshenko beam theory (TBT)

Substituting Eqs. (48)–(51) into Eqs. (35)–(37) lead to the following system of algebraic equations:

\[
\begin{bmatrix}
\alpha^2 A_{xx} & 0 & -x^2 B_{xx} \\
0 & k_1 x^2 A_{az} & -x^2 A_{az} \\
-x^2 B_{xx} & -x^2 k_1 A_{az} & x^2 D_{xx} + k_1 A_{az} \\
\end{bmatrix}
\begin{bmatrix}
U_n \\
W_n \\
\phi_n \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\left( |x^2 (e_0 a)|^2 + 1 \right) \frac{Q_n}{L} \quad (56)
\]

Eq. (56) can be solved to obtain the Fourier coefficients for Timoshenko beam theory

\[ U_n = \frac{B_{xx} |x^2 (e_0 a)|^2 + 1 |Q_n|}{x^2 \left( A_{xx} D_{xx} - B_{xx}^2 \right)} \quad (57) \]

\[ W_n = \frac{\left[ x^2 \left( A_{xx} D_{xx} - B_{xx}^2 \right) + k_1 A_{az} A_{az} \right] |x^2 (e_0 a)|^2 + 1 |Q_n|}{x^2 k_1 A_{az} \left( A_{xx} D_{xx} - B_{xx}^2 \right)} \quad (58) \]

\[ \phi_n = \frac{A_{xx} |x^2 (e_0 a)|^2 + 1 |Q_n|}{x^2 \left( A_{xx} D_{xx} - B_{xx}^2 \right)} \quad (59) \]

Also, note that complete solutions for the displacements are known from Eqs. (48)–(50).

5.1.2. Euler–Bernoulli beam theory (EBT)

With the similar process, when Eqs. (48), (49) and (51) into Eqs. (46) and (47), one obtains the following system of algebraic equations:

\[
\begin{bmatrix}
\alpha^2 A_{xx} & -x^2 B_{xx} \\
-x^2 B_{xx} & x^4 D_{xx} \\
\end{bmatrix}
\begin{bmatrix}
U_n \\
W_n \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\left( |x^2 (e_0 a)|^2 + 1 \right) \frac{Q_n}{L} \quad (60)
\]

Eq. (56) can be solved to obtain the Fourier coefficients for Euler–Bernoulli beam theory

\[ U_n = \frac{B_{xx} |x^2 (e_0 a)|^2 + 1 |Q_n|}{x^2 \left( A_{xx} D_{xx} - B_{xx}^2 \right)} \quad (61) \]

\[ W_n = \frac{A_{xx} |x^2 (e_0 a)|^2 + 1 |Q_n|}{x^4 \left( A_{xx} D_{xx} - B_{xx}^2 \right)} \quad (62) \]

5.2. Buckling problem

5.2.1. Timoshenko beam theory (TBT)

For the buckling problem, the external loads \(q\) and \(f\) are set to zero. By substituting Eqs. (48)–(50) into the governing equations given by Eqs. (35)–(37), one obtains:

\[
\begin{bmatrix}
\alpha^2 A_{xx} & 0 & -x^2 B_{xx} \\
0 & x^2 k_1 A_{az} - 2x^2 f - x^2 (e_0 a)^2 f & -x^2 k_1 A_{az} \\
-x^2 B_{xx} & -x^2 k_1 A_{az} & x^2 D_{xx} + k_1 A_{az} \\
\end{bmatrix}
\begin{bmatrix}
U_n \\
W_n \\
\phi_n \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix} \quad (63)
\]

Eq. (63) defines a standard eigenvalue problem with \(i_n = \frac{P}{P_c}(n)\). The eigenvalues \(i_n\) are found by setting the determinant of the coefficient matrix in Eq. (63) to zero. The smallest eigenvalue \(i_1 = \frac{P}{P_c}(n)\) is the critical buckling load. After some mathematical manipulations, the exact formula of the critical buckling load for Timoshenko beam theory is obtained as:

\[ P = P_c(n) = \frac{x^2 k_1 A_{az} \left( A_{xx} D_{xx} - B_{xx}^2 \right)}{|x^2 (e_0 a)^2 + 1 |x^2 \left( A_{xx} D_{xx} - B_{xx}^2 \right)| + k_1 A_{az} A_{az}} \quad (64) \]

5.2.2. Euler–Bernoulli beam theory (EBT)

For Euler–Bernoulli beam theory, an eigenvalue equation and the exact formula of the critical buckling load are obtained, respectively

<table>
<thead>
<tr>
<th>( \frac{1}{L} )</th>
<th>( \mu \times (e_0 a)^2 )</th>
<th>EBT</th>
<th>TBT</th>
</tr>
</thead>
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<tr>
<td>Ref. [27]</td>
<td>Present</td>
<td>Ref. [27]</td>
<td>Present</td>
</tr>
<tr>
<td>10</td>
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<td>1.3020</td>
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<tr>
<td>4</td>
<td>1.3347</td>
<td>1.3220</td>
<td>1.3362</td>
</tr>
</tbody>
</table>
6. Numerical results

In this section, various numerical examples are presented and discussed to verify the accuracy of the present nonlocal beam models in predicting the static and stability responses of FG nanobeam. The FG nanobeam has the following material properties: $E_1 = 1$ TPa, $E_2 = 0.25$ TPa, $m_1 = m_2 = 0.3$. The shear correction factor is taken as $k_s = 5/6$ for Timoshenko beam theory. A conservative estimate of the nonlocal parameter $0 < e_0 a < 2$ nm for single-walled carbon nanotubes (SWCNTs) is proposed by Wang [84]. Therefore, in this study, the nonlocal parameter is taken as $e_0 a = 0, 0.5, 1, 1.5, 2$ nm to investigate nonlocal effects on the responses of FG nanobeam. For convenience, the following dimensionless quantities are defined:

$$ \begin{align*}
\left( \begin{array}{c}
\frac{x^2 A_{xx}}{-x^3 B_{xx}} \\
-x^3 B_{xx}
\end{array} \right) &= \left( \begin{array}{c}
U_n \\
W_n
\end{array} \right) = \left( \begin{array}{c} 0 \\
0
\end{array} \right) \\
\mathcal{P} &= P_{cr}(n) = \frac{x^2 \left( A_{xx} D_{xx} - B_{xx}^2 \right)}{A_{xx}[2(e_0 a)^2 + 1]} 
\end{align*} $$

(65)

(66)

### Table 2
Comparison of dimensionless critical buckling load of homogeneous nanobeam.

| $L/h$ | $\mu = (e_0 a)^2$ | EBT Ref. [27] | Present & TBT Ref. [27] | Present |
|-------|------------------|----------------|--------|----------------|--------|
| 10    | 0                | 9.8696         | 9.8696 | 9.6227         | 9.6226 |
|       | 1                | 8.9830         | 8.9830 | 8.7583         | 8.7582 |
| 2     | 2                | 8.2426         | 8.2425 | 8.0364         | 8.0363 |
| 3     | 3                | 7.6149         | 7.6149 | 7.4244         | 7.4244 |
| 4     | 4                | 7.0761         | 7.0760 | 6.8990         | 6.8990 |
| 20    | 0                | 9.8696         | 9.8696 | 9.0657         | 9.0660 |
| 4     | 4                | 8.9830         | 8.9830 | 8.9258         | 8.9257 |
| 50    | 0                | 9.8696         | 9.8696 | 8.8595         | 8.8594 |
|       | 1                | 9.8308         | 9.8307 | 8.8207         | 8.8207 |
| 2     | 2                | 9.7923         | 9.7922 | 8.7822         | 8.7822 |
| 3     | 3                | 9.7541         | 9.7540 | 8.7440         | 8.7440 |
| 4     | 4                | 9.7161         | 9.7161 | 8.7062         | 8.7062 |

Fig. 3. Effect of the aspect ratio on (a) dimensionless deflection for uniform load and (b) dimensionless buckling load for $k = 1$, $e_0 a = 1$ nm.

Fig. 4. Effect of the nonlocal parameter on (a) dimensionless deflection for uniform load and (b) dimensionless buckling load for $k = 1$.
Dimensionless transverse deflection:

\[ w = 100w \frac{E_1I}{q_0L^4} \] 
for uniform load \((67)\)

Dimensionless buckling load:

\[ P_{cr} = \frac{p_{cr}L^2}{E_1I} \] 
\((68)\)

6.1. Verification of the present results: bending and buckling of homogeneous nanobeams

In order to demonstrate the efficiency and accuracy of the present exact solution, some illustrative examples are solved and the present results are compared with the existing data available in the literature. For this purpose, dimensionless deflection of a homogeneous nanobeam subjected to uniformly distributed load and dimensionless buckling load obtained by the present solution are compared with those of Aydogdu [27]. In Tables 1 and 2, results are given for various values of the aspect ratio \(L/h = 10, 20, 50\) and the nonlocal parameter \((\mu = (e_0a)^2) = 0, 1, 2, 3, 4 \text{ nm}^2\). Table 2 clearly shows that the present buckling loads agree very well with the solutions of Aydogdu [27]. However, as seen from Table 1, there are some discrepancies between the deflections for uniform load. On the other hand, the present results seem to be more acceptable. For example, it is known from the course of strength of materials that for the classical EBT \((e_0a = 0)\), the maximum mid-point deflection of a simply-supported beam subjected to uniformly load is \(w_{\text{max}} = \frac{5q_0L^4}{384EI}\). If this value is substituted into Eq. \((67)\), the dimensionless deflection is obtained \(w = \frac{500}{384} = 1.3020\) for all values of the aspect ratio, as given by the present results in Table 1.

6.2. Results for bending and buckling responses

Fig. 3 shows the variation of static and buckling responses of FG nanobeam with the aspect ratio. The local and nonlocal results are given for \(e_0a = 0\) and \(e_0a = 1\) nm, respectively. The gradient index is assumed to be constant, \(k = 1\). In this example, the aspect ratio varies from \(L/h = 10\) to \(L/h = 50\). It is seen from this figure that deflections predicted by the nonlocal theory are larger than those of the local (classical) results whereas the nonlocal solution of the buckling load is smaller than the local buckling load due to the small scale effects. This result indicates that the effect of small scale parameter softens the nanobeam. An interesting result deduced from Fig. 3 is that deflection and buckling load are independent of the aspect ratio for the case of local EBT. On the other hand, with the consideration of the nonlocal parameter, all responses of EBT become dependent on the aspect ratio. It is known that the dependency of the responses on the aspect ratio for local TBT is uniquely due to the effect of shear deformation. Further, as seen

<table>
<thead>
<tr>
<th>(L/h)</th>
<th>(k)</th>
<th>(e_0a) (nm)</th>
<th>Nonlocal parameter, (e_0a) (nm)</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>0.5</td>
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</tr>
<tr>
<td>EBT</td>
<td>TBT</td>
<td>EBT</td>
<td>TBT</td>
</tr>
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<td>5.3383</td>
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<tr>
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<td>1.5453</td>
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</table>
from blue\(^1\) and black dashed lines in Fig. 3, this dependency becomes strong with the effect of the small scale effect. The difference between the solutions of EBT and TBT becomes important as the aspect ratio decreases.

The effect of the nonlocal parameter on the static and buckling responses of FG nanobeam is demonstrated in Fig. 4. The results in this figure are obtained by using nonlocal TBT for the constant value of \( k = 1 \). This figure shows that the responses vary nonlinearly with the nonlocal parameter. The most important observation from the figure is due to the fact that all responses of FG nanobeam with lower aspect ratios (i.e., \( L/d = 10 \)) are strongly affected by the nonlocal parameter than those of the FG nanobeam with relatively higher aspect ratios. From this conclusion obtained above, it can be said that modeling based on the local (classical) beam models is not suitable, and the nonlocal beam models may offer an adequate approximation for the nano-sized structures.

Fig. 5 presents the effect of the gradient index on the dimensionless static deformation and buckling load of FG nanobeam with \( L/h = 10 \) for various values of the nonlocal parameter. One can observe that the dimensionless static deflections decrease and the dimensionless buckling load increases as the gradient index increases. This is due to the fact that an increase in the gradient index yields an increase in the stiffness of the FG nanobeam. There is an abrupt change in the responses when the gradient index varies from 0 to 5, but after passing \( k = 5 \) all of the curves become flatter.

Tables 3 and 4 demonstrate the dimensionless static deflections and the dimensionless buckling load for various values of the nonlocal parameter, aspect ratio and the gradient index. The results deduced from the above figures are also drawn from these tables. However, since there is no reported work for the bending and buckling of FG nanobeams as far as the authors know, it is believed that the tabulated results will be a reference with which other researchers can compare their results.

### 7. Conclusions

Bending and buckling of the FG nanobeam are examined based on the nonlocal Timoshenko and Euler–Bernoulli beam theory. The governing equations and the related boundary conditions are derived using the principal of the minimum total potential energy. The Navier-type solution is developed for simply-supported boundary conditions, and exact formulas are proposed for the deflections and the buckling load. The effects of nonlocal parameter, aspect ratio, various material compositions on the static and stability responses of the FG nanobeam are discussed. Numerical results show that the nonlocal effects play an important role on the static and buckling responses of the FG nanobeam. The new nonlocal beam model produces larger deflection and smaller buckling load than the classical (local) beam model. Therefore, the small scale effects (or nonlocal effects) should be considered in the analysis of mechanical behavior of nanostructures. Also, note that reasonable choice of the value of the nonlocal parameter is also crucial to ensure the validity of the nonlocal beam models. Further, it is found that the power-law exponent has a great influence on the responses of FG nanobeam, and the responses can be controlled by choosing proper values of the power-law exponent.

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Simsek M. Cansiz S. Dynamics of elastically connected double-functionally graded beam systems with different boundary conditions under action of a moving harmonic load. Comp Struct 2012;94:2861–78.